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APPLICATIONS OF DYNAMIC PROGRAMMING TO  
SPACE GUIDANCE, SATELLITES, AND  
TRAJECTORIES

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### SUMMARY

The feasibility of space travel and man-made satellites has triggered a rash of interest in the determination of optimal trajectories and generally in guidance and control processes. These problems, for so long of purely mathematical and astronomical concern, have now become part of the engineering domain. The result is that there is a great demand for feasible numerical solutions of the associated analytic problems.

Many of these are classically of great difficulty. As a result of the intensive study of these questions, it is now well appreciated that the classical techniques of the calculus of variations are inoperative unless the problem is rather carefully selected.

In this paper we wish to sketch the applicability of a new mathematical technique, based on the theory of dynamic programming, to the computational solution of trajectory problems. Many problems, seemingly inaccessible to the conventional methods of the calculus of variations, have already been resolved.

APPLICATIONS OF DYNAMIC PROGRAMMING TO SPACE GUIDANCE,  
SATELLITES, AND TRAJECTORIESRichard Bellman  
Stuart Dreyfus  
Robert Kalaba1. Introduction

The feasibility of space travel and man-made satellites has triggered a rash of interest in the determination of optimal trajectories and generally in guidance and control processes. These problems, for so long of purely mathematical and astronomical concern, have now become part of the engineering domain. The result is that there is a great demand for feasible numerical solutions of the associated analytic problems.

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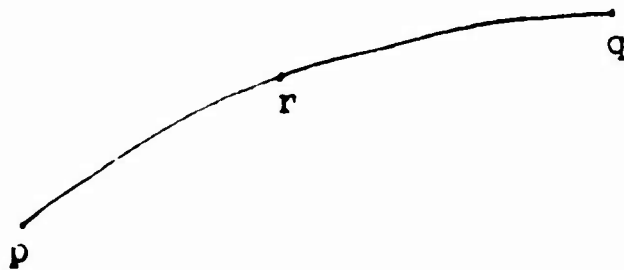
In this paper we wish to sketch the applicability of a new mathematical technique, based on the theory of dynamic programming [1], [2], to the computational solution of trajectory problems. Many problems, seemingly inaccessible to the conventional methods of the calculus of variations, have already been resolved, [2,4,5,6,10,14].

Naturally, there are still a number of major difficulties to be overcome before it can be asserted that we possess a routine approach to realistic processes. What is important is

that we have a general method for attacking guidance and control processes which permits us to study more complex physical problems using quite elementary mathematical concepts.

## 2. A Property of Optimal Time Trajectories

Consider the problem of going from a point  $p$  in phase space to a point  $q$  in phase space. Symbolically, let this be represented in the figure below, where  $r$  is an intermediate point.



Let some particular trajectory be pursued, and denote by  $t(p,q)$  the time required to traverse the path between two generic points. Then, clearly

$$(1) \quad t(p,q) = t(p,r) + t(r,q).$$

Consider now the problem of determining, subject to various constraints on the allowable motions, the path of minimum time between  $p$  and  $q$ . Then, again clearly, if  $r$  is a point on an optimal trajectory the times between  $p$  and  $r$  and  $r$  and  $q$  must also be minimum times for the respective points.

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The problem we face is that of starting out from  $p$ . If we do not know the optimal trajectory, a priori, the foregoing bit of information concerning geodesics is not particularly enlightening.

We can, however, make good use of the following part of this information. Wherever  $r$  is in phase space, and whatever the time consumed going from  $p$  to  $r$ , the time involved in going from  $r$  to  $q$  must be a minimum for this part of the path.

This statement is a particular case of the "principle of optimality" [1].

### 3. Dynamic Programming Approach [3], [4]

It remains to express the foregoing ideas in analytic form. Let

- (1)  $f(p)$  = the time consumed in going from an arbitrary point  $p$  to a fixed point  $q$ , using an optimal path.

Then, the principle of optimality yields firstly the equation

$$(2) \quad f(p) \leq t(p,r) + f(r),$$

for any path  $\overline{prq}$ . To determine  $r$ , we minimize over all admissible choices of  $r$ , obtaining the basic equation

$$(3) \quad f(p) = \min_r [t(p,r) + f(r)].$$

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This equation is the basis for the computational solution of trajectory problems.

Similar considerations lead to the determination of k-th best paths, as is discussed in [7].

#### 4. Brachistochrone

As an example, let us consider the classical brachistochrone problem. Given two points, P and Q, in a vertical plane, we wish to determine a curve connecting these two points with the property that a particle sliding down this curve, solely under the influence of gravity and with no frictional forces, will traverse the path in minimum time.

Take P to the origin and let Q be the fixed point  $(x_0, y_0)$ , as indicated below.

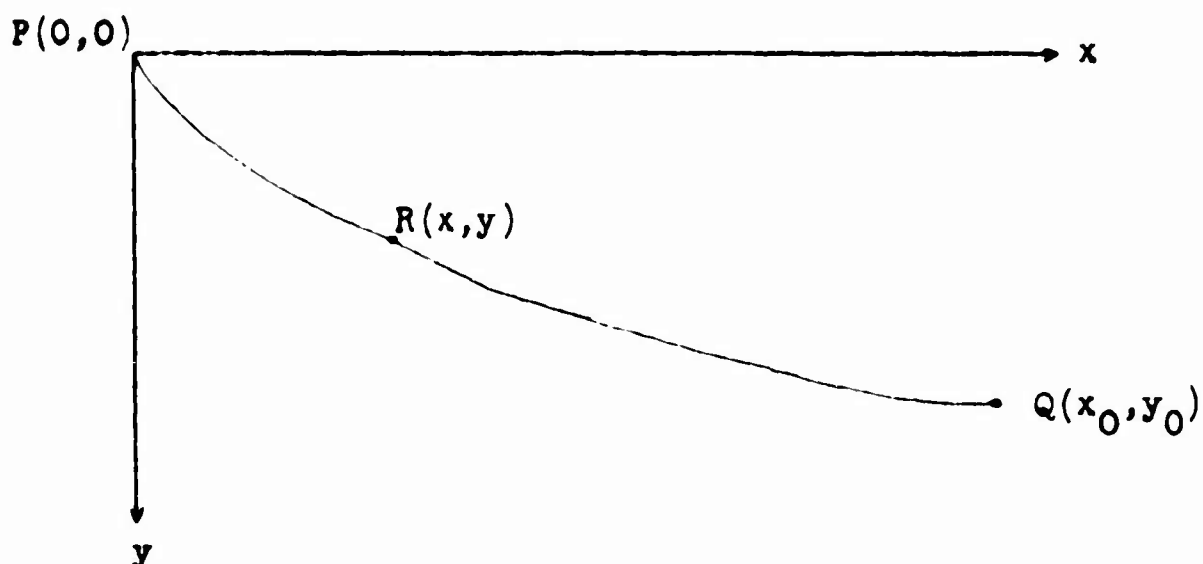
Omitting the gravitational constant, of no import, the problem in the calculus of variations is that of minimizing the integral

$$(1) \quad J(y) = \int_0^{x_0} \left( \frac{1 + y'^2}{y} \right)^{1/2} dx$$

over all curves satisfying the end conditions  $y(0) = 0$ ,  $y(x_0) = y_0$ . In place of treating this in the usual fashion, à la Euler equation and so on, let us introduce the function  $f(x, y)$  defined as follows:

$$(2) \quad f(x, y) = \text{the time required for a particle to fall from } (x, y) \text{ to } (x_0, y_0) \text{ along an optimal path.}$$

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Then a particularization of (3.3) yields the functional equation

$$(3) \quad f(x,y) = \min_{y'} \left[ \left( \frac{1 + y'^2}{y} \right)^{1/2} \Delta + f(x + \Delta, y + y'\Delta) + o(\Delta) \right]$$

where  $y'$  represents the slope of a proposed path at  $x$ , and  $\Delta$  is an infinitesimal.

Passing to the limit, we can obtain a partial differential equation for  $f$  which turns out to be equivalent to the Euler equation. But this is not what we wish to do! Instead we subdivide the  $x$  interval into  $N$  parts, where  $N\Delta = x$  and regard  $f(k\Delta, y) = f_k(y)$  as a function of  $y$  for  $0 \leq y \leq y_0$ .

Then (3) becomes

$$(4) \quad f_k(y) = \min_{y'} \left[ \left( \frac{1 + y'^2}{y} \right)^{1/2} \Delta + f_{k+1}(y + y'\Delta) \right],$$

for  $k = 0, 1, 2, \dots, N-1$ , with  $f_N(y) = 0$ . The determination of  $f_k(y)$  is a computational process which we refer to a digital computer. The minimization over initial directions is



carried out not by means of calculus, but by means of a direct search process over allowable slopes.

One consequence of this is that constraints on  $y'$  simplify the computational process, as they should since we have fewer policies to examine. Further details of the computational aspects, which are seldom trivial, will be found in [10].

### 5. Minimum Energy

Suppose that we wish to go from  $p$  to  $q$  in phase space, reverting to the general formulation of §3, at minimum cost in resources, say minimum fuel. Regarding  $t(p,r)$  not as a time, but as a cost for getting from  $p$  to  $r$ , we see that the general equation of (3.3) is still valid.

Generally, we can treat any variational problem in this fashion, providing it has the Markovian property that we have been employing. To use the foregoing method we need merely know that the optimal continuation from any point in phase space depends only upon that point, and not upon the past history of the process.

Thus, not only can general variational problems involving standard functionals of the form

$$(1) \quad J(y) = \int_{x_0}^{x_1} g(x, y, y') dx,$$

be handled but also many classes of implicit variational problems [5], [9] which cannot readily be treated by conventional methods.

## 6. Stochastic and Adaptive Control Processes

Finally, the same general techniques can be used to treat the more complex stochastic and adaptive control processes [8,9,11,12].

## 7. Discussion

The trajectory and feedback control problems sketched in earlier sections represent merely one application, albeit a particularly important one, of the functional equation technique of dynamic programming to the solution of space problems. Problems which lead to multistage decision processes also occur in the areas of design of multistage rockets [14], communication [3], [15], propagation [13], equipment reliability [16], equipment replacement [17], [18], and so on.

The desire for the design of more nearly optimal systems coupled with the inevitable advances in computer technology should bring dynamic programming methodology into a position of increasing importance in the future in the resolution of space optimization problems.

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